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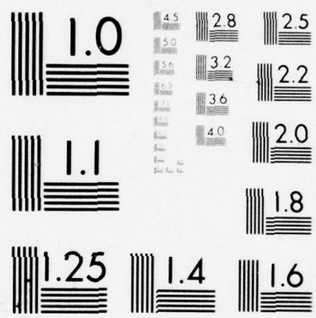

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AN ANNOTATED BIBLIOGRAPHY OF SELECTED REFERENCES ON MAXIMAL INEQUALITIES AND THEIR APPLICATIONS

by

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ABSTRACT

AN ANNOTATED BIBLIOGRAPHY OF SELECTED REFERENCES ON MAXIMAL INEQUALITIES AND THEIR APPLICATIONS

This report provides an annotated bibliography of selected papers, new and old, which we regard as useful to current inquiry into maximal inequalities and their applications. By "maximal inequalities" we mean inequalities placing upper bounds on either the moments or the exceedance probabilities relative to the maximum consecutive sum $\sum_1^k X_i$, $1 \le k \le n$, over a sequence of random variables $X_1,\cdots,X_n$. The bibliography of papers is prefaced by a brief introduction to maximal inequalities and their applications. Various classical results are cited.

0. Summary. Section 1 provides a brief introduction to maximal inequalities and their applications. Various classical results are cited. Section 2 provides an annotated bibliography of selected papers, new and old, which we regard as useful to current inquiry into this area of probability theory. Finally, the list of references at the end includes not only the works cited in Sections 1 and 2 but other key sources as well. Here each item is classified according to emphasis on theory, or applications, or both.

1. Introduction. Let $X_1, X_2 \cdots$ be a sequence of random variables defined on a probability space (Ω, A, P) and pur $S_n = \sum_1^n X_i$, $n = 1, 2, \cdots$. Among the tools most useful to probability theory are the many inequalities which place bounds on the moments

$$E\{\max_{1\le k\le n} |S_k|^{\nu}\},$$

or on the exceedence probabilities

$$P\{\max_{1\le k\le n} |S_k| \ge \lambda\},$$

for suitable choices of ν or λ. Such inequalities are called *maximal* inequalities. A classical example is the following (Tucker (1971)).

KOLMOGOROV'S INEQUALITY. *Let* $X_1, \cdots, X_n$ *be mutually independent with means* 0 *and variances* $\sigma_1^2, \cdots, \sigma_n^2$. *Then*

$$P\{\max_{1\le k\le n} |S_k| \ge \lambda\} \le \lambda^{-2} \sum_{i=1}^{n} \sigma_i^2, \text{ every } \lambda > 0.$$

If, in addition, there exists $K < \infty$ *for which* $P\{|X_i| \le K\} = 1$, $1 \le i \le n$, *then*

$$P\{\max_{1\le k\le n} |S_k| \ge \lambda\} \ge 1 - \frac{(\lambda+K)^2}{\sum_1^n \sigma_i^2 + (\lambda+K)^2 - \lambda^2}, \text{ every } \lambda>0.$$

These two inequalities are extremely useful in proving strong laws of large numbers for independent variables. They are also applied in proving convergence and stability properties for consecutive sums of independent random variables (Lòeve (1963)). The following theorem developed by Kolmogorov is such a result.

THREE SERIES CRITERION. *Let* $X_1, X_2, \cdots$ *be independent. The series* $\sum_1^\infty X_n$ *converges almost surely if and only if for some constant* $C > 0$ *the following three series converge:*

$$\text{(i)} \quad \sum_1^\infty P\{|X_n| \geq C\}$$

$$\text{(ii)} \quad \sum_1^\infty E\{X_n \; I(|X_n| < C)\}$$

$$\text{(iii)} \quad \sum_1^\infty \mathrm{Var}\{X_n \; I(|X_n| < C)\} \quad .$$

The Hàjek-Rényi inequality contains the Kolmogorov inequality as a special case. It was found by Hàjek in 1953 and Rényi later provided the standard proof of this result (Hàjek and Rényi (1955)).

HÀJEK-RÉNYI INEQUALITY. *Let* $\{X_i\}$ *be mutually independent with means* 0 *and finite variances* σ_i^2 . *Let* $\{c_i\}$ *be a nonincreasing sequence of positive numbers. Then, for* $\lambda > 0$ *and for positive integers* $m \leq n$,

$$P\{\max_{m \leq i \leq n} c_i |X_1 + \cdots + X_i| \geq \lambda\} \leq \lambda^{-2}(c_m^2 \sum_{i=1}^m \sigma_i^2 + \sum_{i=m+1}^n c_i^2 \sigma_i^2) \ .$$

The use of this inequality in place of Kolmogorov's inequality makes it possible to simplify the proofs of strong laws of large numbers and related theorems (see Gnedenko (1968)).

Another widely used inequality of the theory of sums of independent random variables is the Lévy inequality. It is of somewhat different nature than the Kolmogorov or Hàjek-Rényi inequalities. By centering sums of

independent random variables at suitable medians, Lévy (Lòeve (1963)) obtains an inequality which can play the role of Kolmogorov's inequality. Let $\mu(X)$ denote a median for the random variable X.

LÉVY INEQUALITY. *Let* $\{X_i\}$ *be mutually independent. Then, for every* $\lambda > 0$,

$$P\{\max_{1\le i\le n} [S_i - \mu(S_i - S_n)] \ge \lambda\} \le 2\, P\{S_n \ge \lambda\}$$

and

$$P\{\max_{1\le i\le n} |S_i - \mu(S_i - S_n)| \ge \lambda\} \le 2\, P\{|S_n| \ge \lambda\} .$$

Random variables which are not necessarily independent will now be considered. In the case of orthogonal random variables, a classical theorem is the Rademacher - Menšov inequality.

RADEMACHER-MENŠOV INEQUALITY. *Assume without loss of generality that the random variables* $X_1,\cdots, X_n$ *have means* 0. *Let* $X_1,\cdots, X_n$ *be mutually orthogonal* (*i.e.*, $E\{X_iX_j\} = 0,\ i \ne j$) *and have variances* $\sigma_1^2,\cdots,\sigma_n^2$. *Then*

$$E\{\max_{1\le i\le n} |S_i|^2\} \le (\log_2 4n)^2 \sum_{i=1}^{n} \sigma_i^2 .$$

Playing a role similar to Kolmogorov's inequality in the independent case, this inequality is applied in the proofs of almost sure convergence and stability theorems well-known in the theory of orthogonal functions (see Loève 1963)).

Both Kolmogorov's and Lévy's inequalities can be generalized by examination of their proofs and noting that the centering of the independent random variables at expectations or medians was a key. Propositions similar to those in the case of independence are obtained by means of centering at conditional expectations. Such centerings provide an important dependency model, martingales,

which is a natural generalization of that of consecutive sums of independent random variables centered at expectations (Loève (1963)). Doob (1953) considers a more general setting and establishes the following inequality.

DOOB INEQUALITY. *Let* $X_1,\cdots, X_n$ *be a submartingale. Then, for all* $\lambda > 0$,

$$P\{\max_{1\le i\le n} X_i \ge \lambda\} \le \lambda^{-1} E\{I(\max_{1\le i\le n} X_i \ge \lambda)X_n\} \le \lambda^{-1} E\{|X_n|\} .$$

Furthermore, if the X_i's *are nonnegative, then*

$$E\{\max_{1\le j\le n} X_j^\nu\} \le \begin{cases} \frac{e}{e-1} + \frac{e}{e-1} E\{X_n \log^+ X_n\}, & \text{for } \nu = 1 \\ (\frac{\nu}{\nu-1})^\nu E\{X_n^{\ \nu}\}, & \text{for } \nu > 1. \end{cases}$$

This theorem contains Kolmogorov's inequality which provided the main tools for the proofs of strong laws of large numbers. An examination of these proofs reveals that the assumed independence of these variables was used only to derive certain inequalities among expectations. Hence, by the above theorem, the main results in the independence case carry over to martingales and submartingales. Such generalizations are important for many applications and they throw new light on the nature of the strong laws of large numbers. The limit properties of submartingales are summarized in the Submartingale Convergence Theorem found in Loève (1963). The proof of this result is based on Doob's inequality.

A powerful technique for proving the weak convergence of a random sequence in C[0,1] is to first prove that the finite dimensional distributions of the sequence converge weakly and then to show that the sequence is relatively compact. In order to use this method, an effective criterion for relative compactness is needed. Prohorov's theorem states that in an arbitrary metric

space a sequence of measures is relatively compact if it is tight. Thus the necessary criterion now is to show tightness of a sequence. Billingsley (1968) establishes tightness for sequences of random elements of $C[0,1]$ by finding bounds for the distribution of the maximum of certain partial sums. The following maximal inequality, proved by Billingsley (1968, p. 94), leads to a practical tightness criterion.

Let $X_1, \cdots, X_n$ *be arbitrary random variables. Suppose that for constants* $\nu \geq 0$ *and* $\gamma > 1$ *and nonnegative constants* $u_1, \cdots, u_n$,

$$P\left(\left|\sum_{k=i}^{j} X_k\right| \geq \lambda\right) \leq \lambda^{-\nu} \left(\sum_{k=i}^{j} u_k\right)^{\gamma} \qquad (\text{all } 1 \leq i \leq j \leq n, \text{ all } \lambda > 0).$$

Then

$$P\left(\max_{1 \leq m \leq n} \left|\sum_{k=1}^{m} X_k\right| \geq \lambda\right) \leq C_{\nu,\gamma} \left(\sum_{k=1}^{n} u_k\right)^{\gamma} \qquad (\text{all } \lambda > 0),$$

where $C_{\nu,\gamma}$ *depends only on* ν *and* γ.

A Hàjek-Rényi type inequality for U-statistics obtained by Sen (1970) is used to show the weak convergence of a sequence of random variables $\{Y_N(t)\}$ to a Brownian bridge. It is applied in the following manner. First a related sequence $\{Y_N^\circ(t)\}$ is shown to converge in distribution, as $N \to \infty$, to a Brownian bridge. The maximal inequality is then used to show $\sup_t |Y_N(t) - Y_N^\circ(t)|$ converges in law to 0 as $N \to \infty$. Hence the convergence of $\{Y_N(t)\}$ is shown. Other forms of the Hàjek-Rényi inequality for U-statistics are obtained in Serfling (1974).

Other uses of maximal inequalities pertain to rates of convergence in strong laws of large numbers and to laws of the iterated logarithm (see Serfling, 1970b). Dudley (1972) adapts a maximal inequality of Skorohod to

obtain speeds of metric probability convergence. Simultaneous confidence regions for stochastic vectors are obtained by Sen (1971) by using a Hàjek-Rényi - type inequality. There are many other applications of maximal inequalities but the above discussion more than substantiates their importance.

2. Annotated Bibliography.

ALEXITS, G. (1973). On the convergence of function series. *Acta. Sci. Math. Szeged* 34 1-9.

The author proves an inequality which plays a role similar to the role played by the Rademacher-Menšov inequality in the theory of orthogonal series.

Let Ω *be a measurable space with a positive measure* μ *and* $\{f_n(x)\}$ *a sequence of measurable functions in* Ω. *On the measurable set* $E \subset \Omega$, *consider the Lebesgue functions of the system* $\{f_n(x)\}$,

$$L_n(x) = \int_E \Big| \sum_{k=0}^{n} f_k(x) f_k(y) \Big| \, d\mu(y) ,$$

and for an index sequence $\nu_1 < \nu_2 < \cdots$ *set*

$$L_{\nu_n}(E) = \int_E \max_{0 \le j \le n} L_{\nu_j}(x) \, d\mu(x) .$$

Let $m \le n$ *be fixed positive integers and denote by* $m(x)$ *and* $n(x)$ *measurable functions taking only integer values between* m *and* n. *Then*

$$\int_E |S_{\nu_{n(x)}}(x) - S_{\nu_{m(x)}}(x)| \, d\mu(x) \le (16 L_{\nu_n}(E) \sum_{k=\nu_m}^{\nu_n} a_k^2)^{1/2} ,$$

where $\{a_n\}$ *is a sequence of real numbers and*

$$S_{\nu_n}(x) = \sum_{k=0}^{\nu_n} a_k f_k(x) .$$

The above inequality becomes a maximal inequality by choosing for $m(x)$ the least integer greater than or equal to m and for $n(x)$ the largest integer

less than or equal to n such that

$$|S_{\nu_{n(x)}} - S_{\nu_{m(x)}}| = \max_{m \le i \le j \le n} |S_{\nu_j}(x) - S_{\nu_i}(x)| .$$

This inequality is then used to prove the result:

If $\sum_1^\infty a_n^2 < \infty$ *and* $L_{\nu_n}(E) \le K$ $(n = 1, 2, \cdots)$ *then* $\{S_{\nu_n}(x)\}$ *converges almost surely on* E. □

BICKEL, P. J. (1970). A Hàjek-Rényi extension of Levy's inequality and some applications. *Acta Math. Acad. Sci. Hung.* 21 199-206.

Bickel's extension of Levy's inequality is two-fold. He deals with symmetric, independent random variables $\{X_i\}$ and introduces a decreasing sequence of nonnegative constants $\{c_i\}$ into the inequality, as in the Hàjek-Rényi inequality.

Let $\{X_i\}$ *be an independent sequence of random variables symmetrically distributed about zero and let* $\{c_i\}$ *be a decreasing sequence of constants. Let* g *be a nonnegative convex function on the reals.*
Define

$$G_n = \sum_{i=1}^{n-1} (c_i - c_{i-1})\, g(S_i) + c_n\, g(S_n) ,$$

where $S_n = \sum_1^n X_i$. *Then, for all* $\varepsilon > 0$,

$$P\{\max_{1 \le k \le n} c_k\, g(S_k) \ge \varepsilon\} \le 2\, P\{G_n \ge \varepsilon\}.$$

This inequality is used to obtain conditions under which $E\{\sup_{n \ge 1} c_n |S_n|^\alpha\} < \infty$, where $\alpha \ge 1$. This result is useful in optimal stopping problems involving payoffs of the form $c_n |S_n|^\alpha$. □

BIRNBAUM, Z. W. and MARSHALL, A. W. (1961). Some multivariate Chebyshev inequalities with extensions to continuous parameter processes. *Ann. Math. Statist.* 32 687-703.

A model for the various generalizations of Chebyshev's inequality is discussed. A standard proof of these inequalities is then outlined. Examination of this method of proof yields some general techniques concerning the problems of deriving inequalities and of proving sharpness. The authors then derive a maximal inequality and show that equality can be achieved. Several known generalizations of Kolmogorov's inequality are obtained by specifing the variables and further specializing the assumptions in their maximal inequality. □

BOROVKOV, A. A. (1972). Notes on inequalities for sums of independent variables. *Th. Prob. Applic.* 17 556-557.

The author proves that inequalities obtained by Nagaev and Fuk (1971) for the probability $P\{S_n \geq x\}$, where the $S_n = \sum_1^n X_i$ are sums of independent random variables X_i, remain valid also for the maximum of the sums provided one further condition is imposed. The inequalities have the form

$$P\{S_n \geq x\} \leq \sum_{k=1}^{n} P\{X_k \geq y_k\} + P ,$$

where P is an estimate of the probability $P\{\tilde{S}_n \geq x\}$, where $\tilde{S}_n$ is the sum of the truncated variables $\tilde{X}_k = \min(y_k, X_k)$. The author also demonstrates that the estimate of $P\{S_n \geq x\}$ by pseudo-moments obtained by Ebralidge (1971) is preserved, with insignificant changes, for $P\{\max_{k \leq n} S_k \geq x\}$. □

BURKHOLDER, D. L. (1964). Maximal inequalities as necessary conditions for almost everywhere convergence. *Z. Wahrscheinlichkeitstheorie* 3 75-88.

Let $(\Omega, \mathcal{U}, \mu)$ be a probability space. Let D be the collection of all sequences $f = (f_1, f_2, \ldots)$ with each f_n an $\mathcal{U}$-measurable function from Ω into the complex numbers. For $f \in D$, define f^* by

$$f^*(\omega) = \sup_{1 \leq n < \infty} |f_n(\omega)| \ , \ \omega \in \Omega.$$

Let $C \subset D$ and $0 < p < \infty$.

Question: What conditions on C *assure the existence of a real number* K *satisfying*

$$\mu\{f^* > \lambda\} \leq \lambda^{-p} K \ , \ \lambda > 0 \ , \ f \in C \ ?$$

In particular, under what conditions on C does the almost sure convergence of each sequence in C imply the existence of such a K?

Burkholder answers the question in the main theorem of his article. The assumptions and conclusions of the theorem are complicated. Briefly, the class of sequences to which this theorem may be applied must be large, in a certain sense, and closed under certain operations by which new sequences are formed. The author presents examples of sets C that satisfy these conditions. The theorem is then applied to a problem in operator ergodic theory. □

BURKHOLDER. D. L. (1973). Distribution function inequalities for martingales. *Ann. of Prob.* 1 19-42.

This article is a guide to some recent martingale inequalities. The underlying theory in these inequalities was introduced in Burkholder and

Gundy (1970). The author's main objective is to simplify some of the ideas and methods of the earlier article and to illustrate their use by a number of applications, old and new. □

<u>BURKHOLDER, D. L. and GUNDY, R. F.</u> (1970). Extrapolation and interpolation of quasi-linear operators on martingales. *Acta Math.* (*Uppsala*) <u>124</u> 249-304.

From the author's introduction, "In this paper we introduce a new method to obtain one-sided and two-sided integral inequalities for a class of quasi-linear operators. The suitable choice of starting and stopping times for martingales, together with the systematic use of maximal functions and maximal operators, is central to our methods."

Let $f = (f_1, f_2, \ldots)$ be a martingale and $d = (d_1, d_2, \ldots)$ its difference sequence. Let $f^* = \sup_{n \geq 1} |f_n|$. f^* is related to the square function, $S(f) = (\sum_1^\infty d_k^2)^{1/2}$ by

$$c_p \| S(f) \|_p \leq \| f^* \|_p \leq C_p \| S(f) \|_p$$

where c_p and C_p are constants and $1 \leq p \leq \infty$. New information about this inequality is obtained in two directions. For a special class of martingales, the authors' method allows them to extend this inequality to the range $0 < p \leq \infty$. In a second direction, the authors show that the operator $S: f \to S(f)$ may be replaced by other operators in the above inequality. Two applications of these results are in the areas of classical orthogonal series and of Brownian motion processes. More general integral inequalities and other applications are also presented. □

CHOW, Y. S. (1960). A martingale inequality and the law of large numbers. *Proc. Amer. Math Soc.* 11 107-111.

The following extension of the Hàjek-Rényi inequality is presented.

Let $\{Y_k: k \ge 1\}$ *be a semi-martingale; let* $c_1 \ge c_2 \ge \cdots$ *be positive constants and let* $\varepsilon > 0$. *Then*

$$P\{\max_{1\le k\le n} c_k Y_k \ge \varepsilon\} \le \varepsilon^{-1} \left(\sum_{k=1}^{n-1} (c_k - c_{k+1}) E\{Y_k^+\} + c_n E\{Y_n^+\}\right),$$

where $Y^+ = \max(Y, 0)$.

Thus, if Y_k is the square of the sum of k random variables with zero means and finite variances, then the inequality reduces to the Hàjek-Rényi inequality. Chow's inequality contains as a special case an inequality contained in Doob (1953), where $c_k \equiv 1$. The inequality is used to prove a law of large numbers for a martingale. The key to the proof is to apply the inequality in the same manner that Kolmogorov's inequality is applied in proving the law of large numbers for a sequence of independent random variables. □

CSÖRGO MIKLÖS (1968). On the strong law of large numbers and the central limit theorem for martingales. *Trans. Amer. Math. Soc.* 131 259-275.

Generalizations of Kolmogorov's inequality for martingales and for submartingales are presented. The generalizations consist in admitting largely arbitrary factors in Kolmogorov's inequality.

Let $\{Y_k\}$ *be a submartingale with* $Y_k \ge 0$ *for all* k. *If* $\{c_k\}$ *is a non-increasing sequence of positive constants, then for* $\varepsilon > 0$ *and* $m < n$

$$P\{\max_{m\le k\le n} c_k^2 Y_k \ge \varepsilon\} \le \varepsilon^{-1} \left(\sum_{k=m}^{n-1} (c_k^2 - c_{k+1}^2) E\{Y_k\} + c_n^2 E\{Y_n\}\right).$$

The proof of this inequality is similar in style to that of the Hàjek-Rényi inequality, but exploits the submartingale property of the Y_k's to gain the added generality.

The author uses this inequality to prove a generalization of Kolmogorov's inequality for martingales.

Let $\{X_k\}$ *be a sequence of random variables with*

$$\text{(1)} \qquad E\{X_1\} = 0, \qquad E\{X_n | X_1, \cdots, X_{n-1}\} = 0 , \quad n \geq 2 , \text{ w.p.1.}$$

Define $S_n = \sum_1^n X_i$. *If* $E\{S_n^2\} < \infty$ *for all* n *and* $\{c_n\}$ *is an increasing sequence of positive constants, then for all* $m < n$ *and arbitrary* $\varepsilon > 0$

$$P\{ \max_{m \le k \le n} c_k |S_k| \geq \varepsilon \} \leq \varepsilon^{-2} \left(\sum_{k=m}^{n-1} (c_k^2 - c_{k+1}^2) E\{S_k^2\} + c_n^2 E\{S_n^2\} \right)$$

$$= \varepsilon^{-2} \left(c_m^2 \sum_{k=1}^{m} E\{X_k^2\} + \sum_{k=m+1}^{n} c_k^2 E\{X_k^2\} \right) .$$

If $c_k \equiv 1$ and $m = 1$, we have Kolmogorov's inequality for martingales. If the X_k's are independent, we have the Hàjek-Rényi inequality. The proof of this inequality follows by noting that $\{S_n\}$ being a martingale implies $\{S_n^2\}$ is a submartingale and then applying the author's inequality for submartingales. The second form of the bound follows from the "absolute fairness" of the sequence $\{X_k\}$, that is, assumption (1). □

DAVIS, BURGESS (1970). On the integrability of the martingale square function. *Israel J. Math.* 8 187-190.

Let $f = (f_1, f_2, \cdots)$ be a martingale and $\{d_k\}$ be its difference sequence, that is, $f_n = \sum_1^n d_k$, $n \geq 1$. The following result asserts that the L_1-norms of $\sup_{n \ge 1} |f_n|$ and of $(\sum_1^\infty (f_n - f_{n-1})^2)^{1/2}$ are equivalent.

Define $f^* = \sup_{n\geq 1} |f_n|$ *and* $S(f) = (\sum_1^\infty d_k^2)^{1/2}$. *Then there are two positive numbers* a *and* A *such that*

$$a\|S(f)\| \leq \|f^*\| \leq A\|S(f)\|.$$

The author then presents an extension of the above result to a class of more general operators.

Let M *and* N *be matrix operators, that is, an operator which can be expressed in the form*

$$M(f) = (\sum_{j=1}^{\infty} (\limsup_{n\to\infty} |\sum_{k=1}^{n} a_{jk} d_k|)^2)^{1/2} ,$$

where (a_{jk}) *is a matrix of real numbers such that* $a \leq \sum_{j=1}^{\infty} a_{jk}^2 \leq A$, $k \geq 1$, a *and* A *positive numbers. Let* $M_n(f)$ *denote* $M(f^n)$ *where* f^n *is* f *stopped at time* n, *and let* $M^*(f) = \sup_{n\geq 1} M_n(f)$. *There is a constant* $a(M, N)$ *such that if* f *is a martingale then*

$$\|M^*(f)\| \leq a(M, N) \| N^*(f)\| . \qquad \square$$

DUDLEY, R. M. (1972). Speeds of metric probability convergence. *Z. Wahrscheinlichkeitstheorie* 22 323-328.

From the author's introduction, "This is a survey on speeds of convergence in probability limit theorems for the Prohorov metric ρ and the dual-bounded-Lipschitz metric β. Many of the results are new as stated, although they follow rather easily from previously known theorems and proofs."

More specifically, the author investigates speeds of convergence in laws of large numbers, central limit theorems, Glivenko-Cantelli theorems and also for the invariance principle, the convergence to the Poisson process and the convergence of empirical distribution functions. An adaptation of a maximal inequality of Skorohod is proved and used to obtain "big O"

bounds on $\rho(L(Y_n), L(W))$ and $\beta(L(Y_n), L(W))$, where Y_n is the "random broken line" process and W is the Brownian motion process. $\Box$

FRANK, OVE (1966). Generalizations of an inequality of Hàjek-Rényi. *Skand. Akt.* 49 85-89.

The author proves the following theorem.

Let $X_1\ X_2, \cdots$ *be a sequence of nonnegative random variables and let* $A_k = \{X_k \geq \varepsilon\}$, $k = 1, 2, \cdots$, *with* $\varepsilon > 0$. *Define* $X_0 = 0$, $A_0 = \phi$ *and* $B_k = \bar{A}_0\bar{A}_1 \cdots \bar{A}_{k-1}$. *Then*

$$P\{\max_{1\leq k\leq n} X_k \geq \varepsilon\} \leq \varepsilon^{-1} \textstyle\sum_1^n \int_{B_k} (X_k - X_{k-1})dP .$$

From this inequality, the author derives the Doob inequality for submartingales, the Hàjek-Rényi inequality for arbitrary constants and the Rosén (1964) inequality for a random permutation of N real numbers. He applies the inequality in proving an extension of a theorem of Grenander (1965) on almost sure convergence of $S_n/E\{S_n\}$, where $S_n = Y_1 + \cdots + Y_n$, $Y_n = \min(Z_1, \cdots Z_n)$ and $Z_1, \cdots , Z_n$ are nonnegative, independent and identically distributed random variables. $\Box$

HEYDE, C. C. (1968). An extension of the Hàjek-Rényi inequality for the case without moment conditions. *J. Applied Prob.* 5 481-483.
(also see *J. Applied Prob.* 8 430, for correction note).

The Hàjek-Rényi inequality is extended for arbitrary independent random variables $\{X_k\}$ concerning which the existence of expectations and variances is not supposed. This is achieved by writing X_k in the form $X_k = Y_k + Z_k$, where Y_k has zero mean and finite variance. Such a decomposition is always possible, e. g., by means of truncation.

Let $\{c_k\}$ *be a nonincreasing sequence of positive numbers. Then, for any* $\varepsilon > 0$, $0 < \eta < 1$ *and any positive integers* m *and* n ($m < n$)

$$P\{\max_{m\leq k\leq n} c_k|S_k| \geq \varepsilon\} \leq (1-\eta)^{-2}\,\varepsilon^{-2}\,(c_m^2 \textstyle\sum_1^m E\{Y_k^2\} + \sum_{m+1}^n c_k^2\, E\{Y_k^2\} +$$

$$+ \sum_{r+1}^n P\{Z_k \neq 0\} + P\{c_m|\sum_1^r Z_k| \geq 2^{-1}\eta\varepsilon\}\ ,$$

for any $r < m$.

Obviously, the standard Hàjek-Rényi inequality is recoverable from this extended version. Note also that in order to get a sharp inequality, it is necessary to use a decomposition such that the probabilities $P\{Z_k \neq 0\}$ are small. Tomkins (1971) gives a slightly different version of the Heyde extension. □

<u>KOUNIAS, EUSTRATIOS G. and WENG, TENG-SHAN</u> (1969). An inequality and almost sure convergence. *Ann. Math. Statist.* <u>40</u> 1091-1093.

The authors present an inequality for random variables in $L_r, 0 < r \leq 1$, which has the same form as the Hàjek-Rényi inequality.

Let $X_1, X_2, \cdots$ *be a sequence of random variables such that* $E\{|X_i|^r\} = v_i < \infty$ *for some* $0 < r \leq 1$ *and all* $i = 1, 2, \cdots$. *If* $c_1, c_2, \cdots$ *is a non-increasing sequence of positive constants, then for any integers* m, n *with* $m < n$ *and arbitrary* $\varepsilon > 0$

$$P\{\max_{m\leq k\leq n} c_k|X_1 + \cdots + X_k| \geq \varepsilon\} \leq \varepsilon^{-r}(c_m^r \textstyle\sum_1^m E\{|X_i|^r\} + \sum_{m+1}^n c_i^r\, E\{|X_i|^r\}).$$

A second inequality is given for $r \geq 1$. Under the above assumptions, but with $r \geq 1$, the right hand side of the above inequality becomes

$$\varepsilon^{-r}(c_m \textstyle\sum_1^m E^{1/r}\{|X_i|^r\} + \sum_{m+1}^n c_i\, E^{1/r}\{|X_i|^r\})\ .$$

By using the above two inequalities in conjunction with Kronecker's lemma, it is shown that $(\sum_1^n X_i)/b_n \to 0$ almost surely as $n \to \infty$, for any sequence of constants $b_n \uparrow \infty$. The maximal inequalities are then used to show that $\max_{k \geq 1} |\sum_{m+1}^{m+k} X_i| \to 0$ in probability, as $m \to \infty$. This result yields the almost sure convergence of $\sum_1^n X_i$. □

MARSHALL, ALBERT W. (1960). A one-sided analogue of Kolmogorov's inequality. *Ann. Math. Statist.* 31 483-487.

It is shown that for $\varepsilon > 0$ and for a random variable X, with $E\{X\} = 0$ and $E\{X^2\} < \infty$, $P\{X \geq \varepsilon\} \leq E\{X^2\}/(\varepsilon^2 + E\{X^2\})$, for all $\varepsilon > 0$. This paper presents an inequality that generalizes this inequality in the same way that Kolmogorov's inequality generalizes Chebyshev's inequality. An example is presented to show that the inequality is sharp. The proof of the maximal inequality is similar to the standard proof of Kolmogorov's inequality. Marshall's inequality is then extended to continuous parameter martingales. A condition under which equality can be achieved is also given. □

MCLEISH, D. L. (1975). A maximal inequality and dependent strong laws. *Ann. Prob.* 3 829-840.

The author proves an extension of Doob's martingale inequality. This inequality is then used to extend the martingale convergence theorem for L_2-bounded variables and to prove strong laws under dependent assumptions. Strong and ϕ-mixing variables are shown to satisfy the conditions of these theorems and hence strong laws are proved as well for these variables. □

MILLAR, P. WARWICK (1969). Martingales with dependent increments. *Ann. Math. Statist.* 40 1033-1041.

Let $\{A_n\}$ be an increasing family of sub-sigma fields of A. Let $f = (f_n, A_n; n = 1, 2, \cdots)$ be a martingale and define $d_n = f_n - f_{n-1}$. The sequence $g = \{g_n; n = 1, 2, \cdots\}$ is called a transform of f under ν provided

that $g_n = \sum_1^n \nu_k d_k$, where $\nu = \{\nu_n; n = 1, 2, \cdots\}$ is a sequence of random variables with ν_n being A_{n-1}-measurable. L_1-bounds are obtained for $\sup_{1\le k\le n} |g_k|$, where $\{g_k\}$ is a transform of a discrete parameter martingale having independent increments. An application of this result to the theory of stochastic integrals with respect to a continuous parameter martingale with independent increments is presented. ⊓

PETROV, V. V. (1973). On strengthenings of some laws of large numbers for stationary sequences. *Dokl. Akad, Nauk.* 213 42-44 (in Russian).

Petrov formulates the following result on growth of sums of random variables without any supposition on independence or stationarity.

Let $\{X_n\}$ *be a sequence of random variables,* $S_n = \sum_1^n X_i$, *and let* $\{a_n\}$ *be a sequence of nondecreasing numbers with* $a_n \to \infty$ *and* $a_{n+1}/a_n \to 1$. *For* $\beta > 0$ *and fixed* x *from the interval* $(1, 1 + \beta)$, *let there exist positive constants* ε *and* C *such that*

$$P\{\max_{1\le k\le n} S_k \ge x\, a_n\} \le C(\log a_n)^{-1-\varepsilon}$$

for all sufficiently large n. *Then* $\limsup_n S_n/a_n \le 1$ *almost surely.* This result is then used to derive propositions which strengthen some laws of large numbers for stationary sequences. ⊓

PROHOROV, YU. V. (1956). Convergence of random processes and limit theorems in probability theory. *Th. Prob. Applic.* 1 157-212.

Kolmogorov's inequality and a maximal inequality involving exponential bounds are used to prove necessary and sufficient conditions for the convergence of distributions of "random broken lines", formed by sums of independent

random variables, to the distribution of the Wiener process. Their roles in the proof are to show that a sequence of distributions is weakly compact. The inequality:

Let $X_1, \cdots, X_n$ *be independent with* $E\{X_i\} \equiv 0$, $E\{X_i^2\} < \infty$, *all* i, *and* $S_i = X_1 + \cdots + X_i$. *Then, for* $\varepsilon > 0$,

$$P\{\max_{1 \le i \le n} |S_i| > \varepsilon\} \le 2P\{|S_n| > \varepsilon - \sqrt{2\,\mathrm{Var}(S_n)}\},$$

is used to obtain an estimate of the rate of convergence in distribution of a "random broken line 1" process to the Wiener process. □

RAO, B. L. S. PRAKASA (1969). Random central limit theorems for martingales. *Acta Math. Acad. Sci. Hung.* 20 217-222.

The author proves the following result.

Let $\{X_n\}$ *be a strictly stationary ergodic sequence such that* $E\{X_1\} \equiv 0$, $E\{X_1^2\} \equiv 1$, *and* $E\{X_n | X_{n-1}, \cdots, X_1\} = 0$. *Furthermore, assume that* $\{X_n\}$ *satisfies a modified version of the Rosenblatt strong mixing condition. Let* $\{\nu_n\}$ *be any sequence of positive integer valued random variables such that* $n^{-1}\nu_n$ *converges in probability to a positive random variable* μ, *having a discrete distribution. Then*

$$\lim_{n\to\infty} P\{\nu_n^{-1/2}(\textstyle\sum_1^{\nu_n} X_k) \le x\} = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-t^2/2)dt.$$

In the proof of the above theorem, a generalized form of the Kolmogorov inequality is applied to show that $(S_{\nu_n} - S_{[\mu n]})/\sqrt{\mu n}$ converges in probability to 0, where $S_n = X_1 + \cdots + X_n$. □

RÉVÉSZ, P. (1965a). On a problem of Steinhaus. *Acta Math. Acad. Sci. Hung.* 16 311-318.

Let $X_1, X_2, \cdots$ be a sequence of random variables with $\sup_{k\geq 1} E\{X_k^2\} < \infty$. The author establishes the existence of a random variable Y and an increasing sequence $m_1, m_2, \cdots$ of positive integers such that if $c_1, c_2, \cdots$ is a real number sequence with $\sum_1^\infty c_k^2 < \infty$, then the series $\sum_1^\infty c_k (X_{m_k} - Y)$ converges with probability 1. In proving this result, the author derives a maximal inequality analogous to Kolmogorov's inequality. The inequality is used to prove that the sequence $\{S_n\}$ is almost surely Cauchy, where $S_n = \sum_{k=1}^n c_k Z_{n_k}$, $Z_k = X_{m_k} - Y$ and $n_1, n_2, \cdots$ is a sequence of positive integers. □

RÉVÉSZ, P. (1965b). Some remarks on strongly multiplicative systems. *Acta Math. Acad. Sci. Hung.* 16 441-446.

The uniformly bounded sequence of random variables $\{X_i\}$ is called an equinormed strongly multiplicative system (ESMS) if $E\{X_i\} \equiv 0$, $E\{X_i^2\} \equiv 1$ and $E\{X_{i_1}^{r_1} X_{i_2}^{r_2} \cdots X_{i_k}^{r_k}\} = E\{X_{i_1}^{r_1}\} E\{X_{i_2}^{r_2}\} \cdots E\{X_{i_k}^{r_k}\}$ all $1 \leq i_1 < i_2 < \cdots < i_k$, for $k = 1, 2, \cdots$, and $r_j = 1$ or 2, $j = 1, \cdots, k$. The following analogue of the Rademacher-Mensov inequality is derived:

If $\{X_i\}$ *is an* ESMS, *then*

$$E\{\sup_{1\leq j\leq n} (X_1 + X_2 + \cdots + X_j)^4\} = O(n^2 \log^3 n), \quad n\to\infty.$$

This inequality is used to obtain the following law of the iterated logarithm for an ESMS:

If $\{X_i\}$ *is an ESMS, then*

$$P\{\lim_{n\to\infty} \sup(X_1 + \cdots + X_n)/(n \log\log n)^{1/2} \le 6\} = 1. \square$$

<u>RÉVÉSZ, P.</u> (1966). A convergence theorem of orthogonal series. *Acta. Sci. Math. Szeged* <u>27</u> 253-260.

The author proves the following analogue of the Rademacher-Menšov inequality:

If $\{X_i\}$ *is a sequence of random variables for which*

$E\{X_i^4\} \le K < \infty$ *(all* i*)*

(1) and

$$E\{X_i^2 X_j X_k\} = E\{X_i^2 X_j\} = E\{X_i X_j X_k X_\ell\} = E\{X_i X_j X_k\} = E\{X_i X_j\} = E\{X_i\} = 0,$$

where i, j, k *and* ℓ *are distinct and* K *is a positive constant, then*

$$E\{\max_{1\le k\le n} |c_1 X_1 + \cdots + c_k X_k|^4\} \le 8K\,\ell^4(n)\,(\textstyle\sum_1^n c_j^2)^2 ,$$

where $\{c_i\}$ *is an arbitrary sequence of constants and*

$$\ell(x) = \ell_1(x) = \begin{cases} \log x & \text{if } x \ge 2 \\ 1 & \text{if } 0 < x < 2 \end{cases} .$$

The author notes that the above lemma is not the best possible. In Révész (1965b) it was proved that in the case $c_n \equiv 1$, $\ell^4(n)$ can be replaced by $O(1)\ell^3(n)$. The same method can be applied in this more general case to obtain a stronger inequality. But this stronger inequality would not enable the author to obtain a stronger result than the following result, whose proof makes use of the above inequality.

Let $\{X_n\}$ *satisfy* (1). *Let* $\{c_i\}$ *be a sequence of constants and suppose that there exists an integer* r *such that* $\sum_{k=1}^{\infty} c_k^2 \ell_r^2(k) < \infty$, *where* $\ell_r(x)$

is the r-th *iterate of* $\ell(x)$. *Then the series* $\sum_1^\infty c_k X_k$ *is almost surely convergent.* □

RÉVÉSZ, P. (1969). M-mixing systems I. *Acta Math. Acad. Sci. Hung.* 20 431-442.

Under conditions which bound $|E\{X_{i_1} \cdots X_{i_\nu}\}|$ for $1 \leq i_1 < \cdots < i_\nu$, $\nu = 1, 2, 3, 4$, an inequality analogous to the Rademacher-Menšov inequality is proved. The author then uses this inequality to prove an almost sure convergence result for $\sum_1^\infty c_i X_i$, where $\{c_i\}$ is a sequence of constants. □

RÉVÉSZ, P. (1973). A new law of the iterated logarithm for multiplicative system. *Acta. Sci. Math. Szeged* 34 349-358.

The author derives the following maximal inequality:

Let $\{X_i\}$ *be a sequence of uniformly bounded multiplicative random variables. Let* $a_1, \cdots, a_n$ *be a sequence of reals. Let* $S_m = \sum_1^m a_k X_k$, $A^2 = \sum_1^n a_k^2$ *and* K_2, K_3 *be suitable positive constants. Then*

$$P\{\max_{1 \leq m \leq n} |S_m| \geq K_2(A^2 \log \log A)^{1/2}\} \leq K_3 \exp(-2 \log \log A).$$

The key to the proof of the above result is a judicious partition of $1, \cdots, n$ and the use of the inequality: $P\{|S_n| \geq y K_1(2A^2)^{1/2}\} \leq 2 \exp(-y^2)$ for any $y > 0$, where S_n and A are defined as above and $|X_i| \leq K$, all i.

The maximal inequality is then used to prove that, with probability 1, only a finite number of the events

$E_n = (\max_{n_k \leq m \leq n_{k+1}} |S(m) - S(n_k)| \geq \varepsilon K_2 (A^2(n_k) \log \log A(n_k))^{1/2})$ occur, where $\{n_k\}$ is a sequence of positive integers and $A^2(n_k) = \sum_1^{n_k} a_i^2$. This result is

then applied in the proof of a law of the iterated logarithm. □

RÉVÉSZ, P. (1974). A law of the iterated logarithm for weakly multiplicative systems and its application. *Acta Math. Acad. Sci. Hung.* 25 425-433.

The author defines a generalization of multiplicative dependence: Let $\{X_k\}$ be a sequence of uniformly bounded random variables for which there exists a positive constant K_1 such that $|E\{\prod_{i=n}^{m}(1 + b_i X_i)\}| \le K_1$ $(1 \le n < m < \infty)$, where $\{b_i\}$ is any sequence of constants for which $|b_i| \le 1$, all i. For a sequence of random variables $\{X_i\}$ satisfying the above restriction, he obtains the following inequality:

Let $a_1, \cdots, a_n$ *be a sequence of constants; then*

$$P\{\max_{1\le m\le n} |\textstyle\sum_1^m a_k X_k| \ge K_3(A^2 \log\log A)^{1/2}\} \le K_4 \exp(-2 \log\log A),$$

where $A^2 = \sum_1^n a_k^2$, K_3, K_4 *are suitable positive constants.*

Using this maximal inequality, he obtains two laws of the iterated logarithm and shows that these laws generalize some known results concerning subsequences of orthogonal sequences, sequences of lacunary Walsh functions and sequences of martingale differences. □

SEN, PRANAB KUMAR (1970). The Hàjek-Rényi inequality for sampling from a finite population. *Sankhya (Series A)* 32 181-188.

For simple random sampling without replacement from a finite population, the author proves a Hàjek-Rényi type inequality for a broad class of symmetric estimators, namely, Hoeffding's U-statistics. This inequality is then used to provide suitable rates for the almost sure convergence of these statistics. The inequality also yields an inequality comparable to Kolmogorov's inequality

for dependent summands. The inequality is as follows:

Let $X_1, \cdots, X_n$ *represent a random sample of size* n *drawn without replacement from a finite population of size* N. *Define* U_n *as the* U-statistic *corresponding to a kernal* $\phi(X_1, \cdots, X_m)$ *and let* $\theta_N = E\{U_n\}$ *and* $V_n = \text{Var}\{U_n\}$. *Then, for every* $m \le k \le N$ *and every* $\epsilon > 0$,

$$P\{\max_{k\le j\le n} c_j|U_j - \theta_N| > \epsilon\} \le \epsilon^{-2}\left(\sum_{j=k}^{n-1} c_j^2(V_j - V_{j+1}) + c_n^2 V_n\right). \quad \sqcap$$

SEN, PRANAB KUMAR (1972). Finite population sampling and weak convergence to a Brownian bridge. *Sankhga (Series A)* 34 85-90.

The Hàjek-Rényi inequality for U-statistics obtained by Sen (1970) is used to show the convergence of a sequence of functions of U-statistics to a Brownian bridge, for sampling without replacement. $\sqcap$

SERFLING, R. J. (1970a). Moment inequalities for the maximum cumulative sum. *Ann. Math. Statist.* 41 1227-1234.

The theorems of this paper obtain bounds on $E\{M_{a,n}^{\nu}\}$ by assuming appropriate bounds on $E\{|S_{a,n}|^{\nu}\}$, where $S_{a,n} = \sum_{a+1}^{a+n} X_i$ and $M_{a,n} = \max(|S_{a,1}|, \cdots, |S_{a,n}|)$. The only dependence restrictions are those implied by the assumed bound on $E(|S_{a,n}|^{\nu})$. In Theorem A, the bounds may involve parameters of the joint distribution of $X_{a+1}, \cdots, X_{a+n}$, a flexibility particularly useful with non-identically distributed variables. The inequality due to Rademacher-Mensov for $\nu = 2$ and orthogonal X_i's is generalized by Theorem A to $\nu \ge 2$ and other types of dependent random variables. Theorem B provides a bound for $E\{M_{a,n}^{\nu}\}$ which is asymptotically optimal in the sense that it is of the same order of magnitude as the bound assumed for $E\{|S_{a,n}|^{\nu}\}$. Roughly speaking, a factor $(\log_2 2n)^{\nu}$ occurring in the bound given

by Theorem A becomes eliminated. As a consequence, the scope of useful asymptotic applications becomes greatly enlarged. This gain over Theorem A is achieved at the expense of requiring that the bound assumed on $E\{|S_{a,n}|^{\nu}\}$ be for a value of $\nu > 2$ and be a function depending upon $F_{a,n}$ only through n. The crucial difference between the proofs of the two theorems is that Minkowski's inequality is exploited to obtain Theorem A but must be avoided in Theorem B. $\Box$

<u>SERFLING, R. J.</u> (1970b). Convergence properties of S_n under moment restrictions. *Ann. Math. Statist.* <u>41</u> 1235-1248.

A strong law of large numbers is obtained for random variables satisfying moment restrictions of second order only. A maximal inequality of Serfling (1970a) is used in the proof of this result to obtain a bound on $E\{Y_k^2\}$, where $Y_k = \max(|S_{n_k,1}|, \cdots, |S_{n_k,n_{k+1}-n_k}|)$ and $S_{a,n} = \sum_{a+1}^{a+n} X_i$. This same inequality is then used in the proof of an almost sure convergence result for $\sum_1^{\infty} X_i$, again under moment restrictions of second order only. A second maximal inequality of Serfling (1970a) is then used in conjunction with the Borel-Cantelli Lemma to obtain the desired result. The result does not require boundedness of the X_i's, only uniform boundedness of the νth absolute moments. The conclusion improves as ν increases: This is possible because the maximal inequality does not have a factor $(\log_2 2n)^{\nu}$ in the bound, in contrast with the first mentioned inequality. The maximal inequality is then used to obtain rates of convergence in the SLLN and the LIL. In all these results, the only dependence restrictions involved are those implied by assumed bounds on $E\{|S_{a,n}|^{\nu}\}$. $\Box$

<u>SERFLING, R. J.</u> (1974). Probability inequalities for the sum in sampling

without replacement. *Ann. Statist.* 2 39-48.

Maximal inequlaities are obtained in sampling without replacement from a finite population. These inequalities extend the Kolmogorov and Hàjek-Rényi inequalities. The proofs of these inequalities exploit martingale techniques and make use of Chow's inequality. Applications of these inequalities to obtain other inequalities are presented.

Also, an upper bound for the moment generating function of the sample sum is derived and applied to obtain exponential probability inequalities and related moment inequalities.

Applications considered include optional stopping in sequential sampling and large deviations of simple linear rank statistics. □

STEIGER, W. L. (1969). A best possible Kolmogorov-type inequality for martingales and a characteristic property. *Ann Math. Statist.* 40 764-769.

In this paper, the author derives some Kolmogorov-type inequalities. For example, let B(n) be the class of all martingales $\{S_n\}$ of n partial sums $S_i = \sum_1^i X_j$, $1 \le i \le n$, where $\{X_i\}$ is a sequence of random varialbes satisfying

(a) $E\{X_1\} = 0$, $E\{X_i | X_{i-1}, \cdots, X_1\} = 0$

(b) $|X_i| \le T$, almost surely

and

(c) $E\{X_1^2\} \ne 0$, $E\{X_i^2 | X_{i-1}, \cdots, X_1\} \ne 0$, almost surely, all $1 \le i \le n$.

The author proves:

Let $M_n = \max(S_1, \cdots, S_n)$ *and* $C^2 = E\{X_1^2\} + \sum_2^n E\{X_i^2 | X_{i-1}, \cdots, X_1\}$. *If* $\{S_i\} \in B(n)$, $1 \le i \le n$, $K > 0$ *and* $h(x) > 0$ *is a nondecreasing function, then*

$$P\{M_n \geq tC^2\} \leq E\{h(KS_n)\}/h(Ktb) ,$$

where b *is a positive number such that* $C^2 \geq b$ *almost surely.*

The following inequality yields a best possible upper bound for $P\{M_n \geq tC^2\}$ for $\{S_i\} \varepsilon B(n)$

Let $\{S_i\} \varepsilon B(n)$, $0 < b \leq C^2$ *almost surely and take* $0 \leq d^2 \leq nT^2$ *such that* $C^2 \geq d^2$ *almost surely. Then for all* $t > 0$

$$P\{M_n \geq tC^2\} \leq (d^2/(d^2 + tbT))^{(tbT + d^2)/T^2} \exp(tb/T) .$$

With $b = \sup(x > 0: x \leq C^2$ almost surely), the right hand side of the above is a best possible upper bound in $B(n)$. The proof of the above inequality applies a bound for the moment generating function for S_n which is derived in the paper. □

STEIGER, W. L. (1973). A generalization of Dunnage's inequality. *J. London Math. Soc.* 7 1-4.

The following result is an extensive generalization of a result of Dunnage (1970). It is in a sense similar to Kolmogorov's extension of Chebyshev's inequality.

Let $(X_i, F_i;\ i = 1, 2, \cdots)$ be a martingale difference sequence on a probability space (Ω, F, P), i. e., X_i is F_i- measurable, $F_i \subseteq F_{i+1}$ and $E\{X_{i+1} | F_i\} = 0$ for $i = 1, 2, \cdots$, Let $\{a_i\}$ be a sequence of real numbers and let N be the set of natural numbers. For any $m \varepsilon N$, set $S_m = \sum_1^m a_i X_i$, $\sigma_m^2 = \sum_1^m a_i^2$, $M_m = \max_{1 \leq j \leq n} |S_j|$, $b_{k,n} = E^{1/k}\{|X_n|^k\}$ and $b_k(n) = \max_{1 \leq i \leq n} b_{i,n}$.

The author provides a concise proof of the following rather strong result.

If $E\{X_n^2\} = 1$, $E\{X_n^{2k-1}\} = 0$, $b_{k,n} < \infty$ *for all* k *and* n *and* B *is a positive real number, then for any numbers* t *and* d *for which* $t \geq d \geq \sum_1^n E\{|a_i X_i| | U_i\}$,

where $U_i = \{|a_i X_i| \le B\}$,

$$P\{M_n \ge t\} \le (b_{2k}^2 \sigma_n^2 /B^2)^k + ((dt^{-1})^t e^{t-d})^{1/B} . \square$$

STOUT, WILLIAM F. (1973). Maximal inequalities and the law of the iterated logarithm. *Ann. Prob.* 1 322-328.

The author derives two maximal inequalities. The first inequality yields a bound on the exceedance probability of the sup of a nonnegative supermartingale. One of these inequalities is as follows:

Let $\{X_i\}$ *be an arbitrary sequence of random variables and let* $S_{m,n} = \sum_{m+1}^{m+n} X_i$ *Let* $S_{m,n}$ *be generalized Gaussian with parameter* A_n, *for some positive number* A *and all* $m \ge 0$ *and* $n \ge 1$, i. e., *let* $S_{m,n}$ *satisfy* $E\{\exp(uS_{m,n})\} \le \exp(u^2 An/2)$, *for all real* u. *Then, for each* $\nu > 2$, *there exists a positive constant* K_ν *such that*

$$E\{\max_{1\le i\le n} |S_{m,i}|^\nu\} \le K_\nu (An)^{\nu/2} ,$$

for all $m \ge 0$ *and* $n \ge 1$.

The author then uses the maximal inequality approach of Serfling (1970a) to derive an upper half law of the iterated logarithm for generalized Gaussian random variables. An upper half law of the iterated logarithm for supermartingales is also derived. □

SZYNAL, DOMINK (1973). An extension of the Hàjek-Rényi inequality for one maximum of partial sums. *Ann. Statist.* 1 740-744.

This paper gives an extension of the Hàjek-Rényi inequality and of the Kolmogorov inequality free of moment conditions and of the assumption of independence.

If $\{X_k\}$ *is any sequence of random variables,* $S_n = \sum_1^n X_k$ *and* $\{c_k\}$ *is a non-increasing sequence of positive constants, then, for any positive integers* m, n *with* $m < n$ *and arbitrary* $\varepsilon > 0$, *we have*

$$P\{\max_{m\le k\le n} c_k|S_k| \ge 3\varepsilon\} \le 2(\textstyle\sum_1^m E^{1/s}\{|X_i|^r/((b_m\varepsilon)^r + |X_i|^r)\} +$$

$$+ \textstyle\sum_{m+1}^n E^{1/s}\{|X_i|^r/((b_i\varepsilon)^r + |X_i|^r\})^s ,$$

where $s = 1$ *if* $0 < r \le 1$ *and* $s = r$ *if* $r \ge 1$ *and* $b_i = c_i^{-1}$.

This theorem is a generalization of results of Kounias and Weng (1969). The author presents two examples which elucidate the relation between his theorems and those of Kounias and Weng.

The author then applies these inequalities in the proofs of almost sure convergence and convergence in probability results for $\{S_n\}$. □

TEICHER, HENRY (1967). A dominated ergodic type theorem. *Z. Wahrscheinlichkeitstheorie 8* 113-116.

Teicher gives conditions under which $E\{\sup_{n\ge 1} c_n|\sum_1^n X_i|^r\} < \infty$. He deals with the case $r \ge 2$ and $\{X_i\}$ a sequence of iid random variables. He then gives moment conditions on $\{X_i\}$ and places bounds on $\{c_i\}$ in order to achieve finiteness. In proving the main result, the author derives an inequality similar to the Chow (1960) inequality:

If $\{U_n, F_n\}$ *is a martingale with* $E\{U_n^2\} < \infty$ *and* $\{c_n\}$ *is a positive decreasing sequence of constants, then*

$$P\{\max_{1\le j\le n} c_j|U_j| \ge \lambda\} \le \lambda^{-2} \textstyle\sum_1^n c_j^2 E\{U_j - U_{j-1}\}^2 , \text{ for } \lambda > 0 . \ \square$$

VOLKONSKII, V. A. (1957). A multi-demensional limit theorem for homogeneous Markov chains with a countable number of states. *Th. Prob. Applic.* 2 221-244.

The author proves the following theorem:

Let $X_1, X_2, \cdots$ *be independent, identically distributed random variables and let* $b_n^{-1} \sum_1^n X_j$ *converge in distribution. Then, as* $\varepsilon_n \to 0$, $b_n^{-1} \max_{k < \varepsilon_n n} |\sum_1^k X_i|$ *converges in probability to* 0.

The proof of this result makes use of the Kolmogorov inequality and the Lévy inequality. □

WICHURA, MICHAEL (1969). Inequalities with applications to the weak convergence of random processes with multidimensional time points. *Ann. Math. Statist.* 40 681-687.

Let $D_{j_1,\cdots,j_q}$ $(1 \le j_p \le n_p,\ 1 \le p \le q)$ be independent random variables with zero means and finite variances. Set

$$S_{k_1,\cdots,k_q} = \sum_{1\le p\le q} \sum_{1\le j_p \le k_q} D_{j_1,\cdots,j_q}$$

$M = \max\{|S_{k_1,\cdots,k_q}| : 1 \le k_p \le n_p,\ 1 \le p \le q\}$ and $\sigma^2 = E\{S^2_{n_1,\cdots,n_q}\}$.

It is shown that $E\{M^2\} \le 4^q\sigma^2$ and inequalities for $P\{M > 2^q a\}$ are derived, for instance,

$$P\{M > 2^q a\} \le (1 - \sigma^2/a^2)^{-q}\, P\{|S_{n_1,\cdots,n_q}| > a\}\ ,\ \text{if } \sigma^2 < a^2\ .$$

The proof of Wichura's main theorem is based on a submartingale inequality of Doob. Special cases of this theorem yield Kolmogorov's inequality, a limited Hàjek-Rényi inequality and several other well known maximal inequalities. The theorem is used to establish the weak convergence of certain random functions defined in terms of the partial sums, $S_{k_1,\cdots,k_q}$. □

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(To the right of each reference are symbols. An * indicates that the paper appears in the annotated bibliography. An "A" indicates emphasis on applications; a "T" indicates emphasis on theory. An "R" indicates that the paper or book was mentioned only as a general reference in the introduction or annotated bibliography.)

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20. Abstract

This report provides an annotated bibliography of selected papers, new and old which we regard as useful to current inquiry into maximal inequalities and their applications. ~~By~~ "maximal inequalities" are ~~we mean~~ inequalities placing upper bounds on either the moments or the exceedance probabilities relative to the maximum consecutive sum $\sum_1^k X_i$, $1 \le k \le n$, over a sequence of random variables $X_1, \cdots, X_n$. The bibliography of papers is prefaced by a brief introduction to maximal inequalities and their applications. Various classical results are cited.

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